SYMMETRIC INVARIANT COCYCLES ON THE DUALS OF q-DEFORMATIONS

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ABSTRACT. We prove that for $q \in \mathbb{C}^*$ not a nontrivial root of unity any symmetric invariant 2-cocycle for a completion of $U_q\mathfrak{g}$ is the coboundary of a central element. Equivalently, a Drinfeld twist relating the coproducts on completions of $U_q\mathfrak{g}$ and $U\mathfrak{g}$ is unique up to coboundary of a central element. As an application we show that the spectral triple we defined in an earlier paper for the q-deformation of a compact simple simply connected Lie group G does not depend on any choices up to unitary equivalence.

INTRODUCTION

Let G_q, $q > 0$, be the compact quantum group which is the q-deformation in the sense of Drinfeld and Jimbo of a compact simple simply connected Lie group G. In [12] we constructed a quantum Dirac operator D_q on G_q that defines a biequivariant spectral triple, which is an isospectral deformation of that defined by the Dirac operator D on G. To do this we used an analytic version of a result by Drinfeld, due to Kazhdan and Lusztig. This result, see [13] and references therein, produces what we call a Drinfeld twist $\mathcal{F}$, which is an element in the group von Neumann algebra $W^*(G \times G)$, and an isomorphism $\varphi \colon W^*(G_q) \to W^*(G)$ such that $(\varphi \otimes \varphi)\hat{\Delta}_q = \mathcal{F}\hat{\Delta}\varphi(\cdot)\mathcal{F}^{-1}$ and $(\varphi \otimes \varphi)(\mathcal{R}) = \mathcal{F}_{21}q^t\mathcal{F}^{-1}$, where $\mathcal{R}$ is the universal R-matrix for $U_q\mathfrak{g}$ and $t \in \mathfrak{g} \otimes \mathfrak{g}$ is defined by a suitably normalized ad-invariant symmetric form on $\mathfrak{g}$. Most importantly, the Drinfeld associator Φ_{KZ}, which is defined via monodromy of the KZ-equations, is the coboundary of $\mathcal{F}^{-1}$ with respect to the coproduct on $U_q\mathfrak{g}$.

From the outset D_q and the associated spectral triple depend on the choice of $(\varphi, \mathcal{F})$. In this paper we show that a different choice in fact produces the same spectral triple up to unitary equivalence, see Theorem 6.1. So our construction is as canonical as one could possibly hope for. Everything hinges on a uniqueness result for the Drinfeld twist which we establish in Theorem 5.2. It states that for a fixed φ, any Drinfeld twist has to be of the form $(c \otimes c)\mathcal{F}\hat{\Delta}(c)^{-1}$, where c is a unitary central element of $W^*(G)$. Combining this with the contribution from choosing a different φ, we deduce that any Drinfeld twist is of the form $(u \otimes u)\mathcal{F}\hat{\Delta}(u)^{-1}$ for a unitary $u \in W^*(G)$.

An equivalent form, Theorem 2.1, of the uniqueness result for the Drinfeld twist says that any unitary symmetric G_q-invariant 2-cocycle in $W^*(G_q \times G_q)$, see Section 1 for precise definitions, is the coboundary of a central element. The result is also true for nonunitary cocycles, and in this form it makes sense and is valid for all $q \in \mathbb{C}^*$ not a nontrivial root of unity.

The study of 2-cocycles on duals of compact groups was initiated by Landstad [9] and Wassermann [15]. They showed that cohomology classes of 2-cocycles are in a one-to-one correspondence with full multiplicity ergodic actions on operator algebras. It is expected [15] that any 2-cocycle on $\hat{G}$ sufficiently close to the trivial one is defined by a 2-cocycle on the dual of a maximal torus, but this has been proved only for some low rank groups. In particular, it has been shown that $H^2(\widehat{\mathrm{SU}(2)})$ is trivial. The theory of full multiplicity ergodic actions was extended to compact quantum groups in [2], and the second cohomology was computed for the duals of free orthogonal groups, which

implies that $H^2(\widehat{SU_q(2)})$ is trivial. It would be interesting to see whether the methods of the present paper can be applied to compute $H^2(\widehat{G_q})$ for higher rank groups.

The paper is organized as follows. After a brief introduction to nonabelian cohomology for Hopf algebras, we state our main result, Theorem 2.1. In Section 2 we show that any symmetric invariant 2-cocycle is cohomologous to a cocycle $\mathcal{E}$ satisfying two additional properties; that it acts trivially on the isotypic components of the tensor product of two modules corresponding to the highest and next to highest weights. Our goal then is to show that $\mathcal{E} = 1$. This can easily be done for $SU_q(2)$ because the fusion rules are sufficiently simple. However, for higher rank groups new impetus is needed.

Our approach is motivated by the work of Kazhdan and Lusztig [8], who constructed a comonoid in a completion of the Drinfeld category representing the forgetful functor. Such a comonoid, in the equivalent category of $U_q\mathfrak{g}$-modules, is also implicit in Lusztig's book [10]. In Section 3 we give a self-contained presentation of this comonoid.

In the following section we then show that $\mathcal{E}$ acts on the comonoid, and hence defines a natural transformation from the forgetful functor to itself. Considered as an element of a completion of $U_q\mathfrak{g}$ this transformation is a 1-cochain with coboundary $\mathcal{E}$. Pushing the analysis further we then conclude that $\mathcal{E} = 1$.

In Section 5 we reformulate the main result as a statement about uniqueness of the Drinfeld twist, and in Section 6 we apply this to show that the quantum Dirac operator is uniquely defined up to unitary equivalence.

We end the paper with two appendices. In Appendix A we prove the essentially known result that any group-like element affiliated with $W^*(G)$ belongs to the complexification of G. This is used in the main text to show that the group of central group-like elements of the completion of $U_q\mathfrak{g}$ is isomorphic to the center of G. In Appendix B we provide a short proof of our main result in the formal deformation setting following Drinfeld's arguments for 3-cocycles.

1. COHOMOLOGY OF QUANTUM GROUPS

Let (A, Δ) be a discrete bialgebra in the sense of [13]. Therefore $A \cong \oplus_{\lambda \in \Lambda} \operatorname{End}(V_\lambda)$ as an algebra, and

$$\Delta\colon A \to M(A \otimes A) \cong \prod_{\lambda,\mu} \operatorname{End}(V_\lambda \otimes V_\mu)$$

is a nondegenerate homomorphism satisfying coassociativity and which comes with a counit ε.

Adapting the usual definition of cohomology for Hopf algebras, see e.g. Section 2.3 in [11], define an operator

$$\partial\colon M(A^{\otimes n})^\times \to M(A^{\otimes (n+1)})^\times,$$

where the superindex $\times$ denotes invertible elements, by

$$\partial\chi = (\Delta_0(\chi)\Delta_2(\chi)\dots)(\Delta_1(\chi^{-1})\Delta_3(\chi^{-1})\dots),$$

where $\Delta_i = \iota \otimes \cdots \otimes \iota \otimes \Delta \otimes \iota \otimes \cdots \otimes \iota$ with Δ in the ith position for $0 < i < n+1$, and $\Delta_0(\chi) = 1 \otimes \chi$, $\Delta_{n+1}(\chi) = \chi \otimes 1$. In particular, if $u \in M(A)^\times$ and $\mathcal{E} \in M(A \otimes A)^\times$, then we have

$$\partial u = (u \otimes u)\Delta(u)^{-1}, \quad \partial\mathcal{E} = (1 \otimes \mathcal{E})(\iota \otimes \Delta)(\mathcal{E})(\Delta \otimes \iota)(\mathcal{E}^{-1})(\mathcal{E}^{-1} \otimes 1).$$

An n-cochain $\chi \in M(A^{\otimes n})^\times$ is a called a cocycle if $\partial\chi = 1$, and it is called a coboundary if χ belongs to the image of ∂. In general, an n-coboundary is not necessarily a cocycle, but this is the case for $n = 2$.

Two 2-cochains $\mathcal{E}, \mathcal{F}$ are said to be cohomologous if there exists a 1-cochain u such that

$$\mathcal{E} = (u \otimes u)\mathcal{F}\Delta(u)^{-1}.$$

Then $\partial\mathcal{E} = (u\otimes u\otimes u)\partial\mathcal{F}(u^{-1}\otimes u^{-1}\otimes u^{-1})$, so if $\mathcal{F}$ is a cocycle, then so is $\mathcal{E}$, and we also conclude that $\partial^2 u = 1$, although in general $\partial^2 \neq 1$. The set of cohomology classes of 2-cocycles is denoted by $H^2(A)$; this is just a set, the product of two 2-cocycles is not necessarily a cocycle.

We say that a 2-cocycle $\mathcal{E} \in M(A\otimes A)$ is

- **invariant**, if $[\mathcal{E}, \Delta(a)] = 0$ for all $a \in A$;
- **symmetric**, if A is quasitriangular with R-matrix $\mathcal{R}$ and $\mathcal{R}\mathcal{E} = \mathcal{E}_{21}\mathcal{R}$.

If a 2-cocycle $\mathcal{E}$ is symmetric, then the cohomologous cocycle $\mathcal{F} = (v\otimes v)\mathcal{E}\Delta(v)^{-1}$ is symmetric for any v. On the other hand, if $\mathcal{E}$ is invariant, then $\mathcal{F}$ is not necessarily invariant, but this is the case if v is central.

If A is a discrete $*$-bialgebra, so that A completes to a C*-algebra and Δ is a $*$-homomorphism, it makes more sense to consider only unitary cochains. This may change $H^2(A)$, but as the following lemma shows, at least the notion of cohomologous unitary 2-cochains remains the same.

Lemma 1.1. *Suppose (A, Δ) is a discrete $*$-bialgebra. Consider two unitary 2-cochains $\mathcal{E}, \mathcal{F}$ such that $\mathcal{E} = (u\otimes u)\mathcal{F}\Delta(u)^{-1}$ for a 1-cochain u. Then $\mathcal{E} = (v\otimes v)\mathcal{F}\Delta(v)^{-1}$, where v is the unitary part in the polar decomposition $u = v|u|$.*

Proof. It is sufficient to show that $(|u|\otimes|u|)\mathcal{F} = \mathcal{F}\Delta(|u|)$, or since $|u| = \sqrt{u^*u}$, that

$$(u^*u \otimes u^*u)\mathcal{F} = \mathcal{F}\Delta(u^*u),$$

and this is immediate from

$$1 = \mathcal{E}^*\mathcal{E} = \Delta(u^{-1})^*\mathcal{F}^*(u^* \otimes u^*)(u\otimes u)\mathcal{F}\Delta(u^{-1}).$$

□

For invariant cocycles it makes sense to consider the polar decomposition.

Lemma 1.2. *Suppose (A, Δ) is a (quasitriangular) discrete $*$-bialgebra. Let $\mathcal{E}$ be a (symmetric) invariant 2-cocycle, and $\mathcal{E} = \mathcal{F}|\mathcal{E}|$ be the polar decomposition. Then both $\mathcal{F}$ and $|\mathcal{E}|$ are (symmetric) invariant 2-cocycles.*

Proof. A somewhat more general statement is proved in [13, Proposition 1.3]. Briefly, observe that (symmetric) invariant cocycles form a group which is in addition closed under involution (recall that for quasitriangular $*$-bialgebras we require $\mathcal{R}^* = \mathcal{R}_{21}$). It follows that $\mathcal{E}^*\mathcal{E}$ is again such a cocycle. Taking the square root, we conclude that $|\mathcal{E}|$ and $\mathcal{F} = \mathcal{E}|\mathcal{E}|^{-1}$ are (symmetric) invariant cocycles as well. □

We end this section with a categorical perspective which is convenient to keep in mind. Consider the category $A\text{-Mod}_f$ of finite dimensional nondegenerate A-modules, and let $F\colon A\text{-Mod}_f \to \mathcal{V}ec$ be the forgetful functor. Then $M(A)$ as an algebra is identified with the algebra $\text{Nat}(F)$ of natural transformations from F to itself.

An invertible element $\mathcal{E} \in M(A\otimes A)$ defines a natural isomorphism

$$F_2\colon F(U)\otimes F(V) \xrightarrow{\ \mathcal{E}^{-1}\ } F(U\otimes V).$$

Assume that $(\varepsilon\otimes\iota)(\mathcal{E}) = (\iota\otimes\varepsilon)(\mathcal{E}) = 1$. Then $\mathcal{E}$ is a cocycle if $(F, F_2, F_0 = \iota)$ is a tensor functor, that is, the diagram

$$
\begin{array}{ccc}
F(U)\otimes F(V)\otimes F(W) & \xrightarrow{\ \mathcal{E}^{-1}\otimes 1\ } & F(U\otimes V)\otimes F(W) \\
{\scriptstyle 1\otimes\mathcal{E}^{-1}}\Big\downarrow & & \Big\downarrow{\scriptstyle (\Delta\otimes\iota)(\mathcal{E}^{-1})} \\
F(U)\otimes F(V\otimes W) & \xrightarrow[\ (\iota\otimes\Delta)(\mathcal{E}^{-1})\ ]{} & F(U\otimes V\otimes W)
\end{array}
$$

commutes. (We remark that a 2-cocycle such that $(\varepsilon \otimes \iota)(\mathcal{E}) = (\iota \otimes \varepsilon)(\mathcal{E}) = 1$ is called counital. Any 2-cocycle is cohomologous to a counital one, since by applying ε to the middle term of the cocycle identity we conclude that $(\varepsilon \otimes \iota)(\mathcal{E}) = (\iota \otimes \varepsilon)(\mathcal{E}) = (\varepsilon \otimes \varepsilon)(\mathcal{E})1$. If $\mathcal{E}$ is not counital, to define a tensor functor we just have to put $F_0 = (\varepsilon \otimes \varepsilon)(\mathcal{E})$ instead of $F_0 = \iota$.)

Two 2-cocycles are cohomologous if the corresponding tensor functors are naturally isomorphic.

A tensor structure on F defines a comultiplication on $\mathrm{Nat}(F)$ by $\Delta(a) = F_2^{-1} a F_2$. Then a cocycle $\mathcal{E}$ is invariant if the multiplication on $M(A)$ defined by the corresponding tensor structure coincides with the original one.

Another way of looking at it is to say that $\mathcal{E}$ is invariant if it defines a natural morphism $U \otimes V \to U \otimes V$ in $A\text{-Mod}_f$. Then, if is A quasitriangular with R-matrix $\mathcal{R}$, an invariant cocycle $\mathcal{E}$ is symmetric if it commutes with braiding $\sigma = \Sigma \mathcal{R}$, that is, the diagram

$$
\begin{array}{ccc}
U \otimes V & \xrightarrow{\ \mathcal{E}\ } & U \otimes V \\
\sigma \downarrow & & \downarrow \sigma \\
V \otimes U & \xrightarrow[\ \mathcal{E}\ ]{} & V \otimes U
\end{array}
$$

commutes. We remark that this is not the same as saying that the tensor functor defined by $\mathcal{E}$ is braided.

For a discussion of the Drinfeld associator, which is a 3-cocycle, see Section 5.

2. Main result

Let G be a simply connected simple compact Lie group, $\mathfrak{g}$ its complexified Lie algebra. Denote by $\widehat{\mathbb{C}[G]}$ the discrete bialgebra of matrix coefficients of finite dimensional representations of G with convolution product. It is a quasitriangular discrete $*$-bialgebra with R-matrix $\mathcal{R} = 1$. We write $\mathcal{U}(G)$ instead of $M(\widehat{\mathbb{C}[G]})$. It is the algebra of closed densely defined operators affiliated with the von Neumann algebra $W^*(G)$ of G, and it contains the universal enveloping algebra $U\mathfrak{g}$. We denote by $\hat{\Delta}$ and $\hat{\varepsilon}$ the comultiplication and the counit on $\mathcal{U}(G)$.

Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a system $\{\alpha_1, \ldots, \alpha_r\}$ of simple roots. Let $\omega_1, \ldots, \omega_r$ be the fundamental weights. The weight and root lattices are denoted by P and Q, respectively. Let $(a_{ij})_{1 \le i,j \le r}$ be the Cartan matrix and $d_1, \ldots, d_r$ be the coprime positive integers such that $(d_i a_{ij})_{i,j}$ is symmetric. Define a bilinear form on $\mathfrak{h}^*$ by $(\alpha_i, \alpha_j) = d_i a_{ij}$. Let $h_i \in \mathfrak{h}$ be such that $\alpha_j(h_i) = a_{ij}$. For $\lambda \in P$ we shall write $\lambda(i)$ instead of $\lambda(h_i)$. Therefore $\lambda(1), \ldots, \lambda(r)$ are the coefficients of λ in the basis $\omega_1, \ldots, \omega_r$.

For $q \in \mathbb{C}^*$ not a root of unity consider the quantized universal enveloping algebra $U_q \mathfrak{g}$ generated by elements E_i, F_i, K_i, K_i^{-1}, $1 \le i \le r$, satisfying the relations

$$
K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i, \quad K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j,
$$

$$
E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},
$$

$$
\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} E_i^k E_j E_i^{1-a_{ij}-k} = 0, \quad \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} F_i^k F_j F_i^{1-a_{ij}-k} = 0,
$$

where $\begin{bmatrix} m \\ k \end{bmatrix}_{q_i} = \dfrac{[m]_{q_i}!}{[k]_{q_i}! [m-k]_{q_i}!}$, $[m]_{q_i}! = [m]_{q_i} [m-1]_{q_i} \ldots [1]_{q_i}$, $[n]_{q_i} = \dfrac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$ and $q_i = q^{d_i}$. This is a Hopf algebra with coproduct $\hat{\Delta}_q$ and counit $\hat{\varepsilon}_q$ defined by

$$
\hat{\Delta}_q(K_i) = K_i \otimes K_i, \quad \hat{\Delta}_q(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \hat{\Delta}_q(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,
$$

$$\hat{\varepsilon}_q(E_i) = \hat{\varepsilon}_q(F_i) = 0, \quad \hat{\varepsilon}_q(K_i) = 1.$$

If V is a finite dimensional $U_q\mathfrak{g}$-module and $\lambda \in P$, denote by $V(\lambda)$ the space of vectors $v \in V$ of weight λ, so that $K_i v = q_i^{\lambda(i)} v$ for all i. Recall that V is called admissible if $V = \oplus_{\lambda \in P} V(\lambda)$. Consider the tensor category of finite dimensional admissible $U_q\mathfrak{g}$-modules. It is a semisimple category with simple objects indexed by dominant integral weights $\lambda \in P_+$. For each $\lambda \in P_+$ we fix an irreducible $U_q\mathfrak{g}$-module V_λ with highest weight λ. Denote by $\widehat{\mathbb{C}[G_q]}$ the discrete bialgebra defined by our category, so $\widehat{\mathbb{C}[G_q]} \cong \oplus_{\lambda \in P_+} \mathrm{End}(V_\lambda)$. Denote by $\mathcal{U}(G_q)$ the multiplier algebra $M(\widehat{\mathbb{C}[G_q]})$. We shall write $\mathcal{U}(G_q \times G_q)$ instead of $M(\widehat{\mathbb{C}[G_q]} \otimes \widehat{\mathbb{C}[G_q]})$.

The discrete bialgebra $\widehat{\mathbb{C}[G_q]}$ is quasitriangular. The R-matrix $\mathcal{R}_\hbar$ depends on the choice of $\hbar \in \mathbb{C}$ such that $q = e^{\pi i \hbar}$. From now on we will write q^x instead of $e^{\pi i \hbar x}$, provided the choice of $\hbar$ is clear from the context. The R-matrix $\mathcal{R}_\hbar$ can can be defined by an explicit formula, see e.g. [4, Theorem 8.3.9], but for us it will be enough to remember that it is characterized by the following two properties:

- $\hat{\Delta}_q^{op} = \mathcal{R}_\hbar \hat{\Delta}_q(\cdot) \mathcal{R}_\hbar^{-1}$;
- if U is a module with a lowest weight vector ζ of weight λ, so $\zeta \in U(\lambda)$ and $F_i \zeta = 0$ for all i, and V is a module with a highest weight vector ξ of weight μ, so $\xi \in V(\mu)$ and $E_i \xi = 0$ for all i, then

$$\mathcal{R}_\hbar(\zeta \otimes \xi) = q^{(\lambda, \mu)} \zeta \otimes \xi. \tag{2.1}$$

This indeed characterizes $\mathcal{R}_\hbar$, since if U and V are irreducible then $\zeta \otimes \xi$ is a cyclic vector in $U \otimes V$.

We denote by $\mathcal{C}(\mathfrak{g}, \hbar)$ the braided monoidal category of admissible finite dimensional $U_q\mathfrak{g}$-modules with braiding $\sigma = \Sigma \mathcal{R}_\hbar$.

We are now ready to formulate our main result.

Theorem 2.1. *Assume $q \in \mathbb{C}^*$ is not a nontrivial root of unity. Then any symmetric invariant 2-cocycle $\mathcal{E} \in \mathcal{U}(G_q \times G_q)$ is the coboundary of a central element in $\mathcal{U}(G_q)$.*

We will assume $q \neq 1$, leaving the case $q = 1$, which requires minor, mostly notational, modifications, to the reader.

For each $\mu \in P_+$ fix a highest weight vector $\xi_\mu \in V_\mu$. We identify V_0 with $\mathbb{C}$ so that $\xi_0 = 1$.

For $\mu, \eta \in P_+$, define a morphism

$$T_{\mu,\eta} \colon V_{\mu+\eta} \to V_\mu \otimes V_\eta \quad \text{by} \quad \xi_{\mu+\eta} \mapsto \xi_\mu \otimes \xi_\eta.$$

The image of $T_{\mu,\eta}$ is the irreducible isotypic component of $V_\mu \otimes V_\eta$ with highest weight $\mu + \eta$. Since $\mathcal{E}$ is invariant, the action of $\mathcal{E}$ on this image is by a scalar, so there exists $\mathcal{E}(\mu, \eta) \in \mathbb{C}^*$ such that

$$\mathcal{E} T_{\mu,\eta} = \mathcal{E}(\mu, \eta) T_{\mu,\eta}.$$

Lemma 2.2. *The map $P_+ \times P_+ \to \mathbb{C}^*$, $(\mu, \eta) \mapsto \mathcal{E}(\mu, \eta)$, is a symmetric 2-cocycle, that is,*

$$\mathcal{E}(\mu, \eta) = \mathcal{E}(\eta, \mu) \quad \text{and} \quad \mathcal{E}(\mu, \eta)\mathcal{E}(\mu + \eta, \nu) = \mathcal{E}(\eta, \nu)\mathcal{E}(\mu, \eta + \nu).$$

Proof. That it is a cocycle follows from the identity

$$(T_{\mu,\eta} \otimes \iota) T_{\mu+\eta,\nu} = (\iota \otimes T_{\eta,\nu}) T_{\mu,\eta+\nu}$$

by applying the operator $(\mathcal{E} \otimes 1)(\hat{\Delta}_q \otimes \iota)(\mathcal{E})$ to the left side and the same operator $(1 \otimes \mathcal{E})(\iota \otimes \hat{\Delta}_q)(\mathcal{E})$ to the right side.

To see that the cocycle is symmetric, notice that the braiding σ maps the image of $T_{\mu,\eta}$, which is the irreducible isotypic component with highest weight $\mu + \eta$, onto the image of $T_{\eta,\mu}$ (in fact, one can show that $\sigma T_{\mu,\eta} = q^{(\mu,\eta)} T_{\eta,\mu}$). Since $\sigma \mathcal{E} = \mathcal{E}\sigma$ this gives the result. $\square$

It is well-known that any symmetric 2-cocycle on P_+ is a coboundary, see e.g. [13, Lemma 4.2], that is, there exist $c(\mu) \in \mathbb{C}^*$ such that

$$\mathcal{E}(\mu, \eta) = c(\mu + \eta)c(\mu)^{-1}c(\eta)^{-1}.$$

The numbers $c(\mu)$, $\mu \in P_+$, define an invertible element c in the center of $\mathcal{U}(G_q)$. Then replacing $\mathcal{E}$ by $(c \otimes c)\mathcal{E}\hat{\Delta}_q(c)^{-1}$ we get a new symmetric invariant 2-cocycle which is cohomologous to $\mathcal{E}$ via a central element and is such that the corresponding 2-cocycle on P_+ is trivial. In other words, without loss of generality we may assume that

$$\mathcal{E}(\mu, \eta) = 1 \quad \text{for all} \quad \mu, \eta \in P_+. \tag{2.2}$$

This in particular implies that $\mathcal{E}$ is counital, since $(\hat{\varepsilon}_q \otimes \iota)(\mathcal{E})$ acts on V_μ as multiplication by $\mathcal{E}(0, \mu)$.

Fix i, $1 \le i \le r$. Let $\mu, \eta \in P_+$ be such that $\mu(i), \eta(i) \ge 1$. Then the space $(V_\mu \otimes V_\eta)(\mu + \eta - \alpha_i)$ is 2-dimensional, spanned by $F_i\xi_\mu \otimes \xi_\eta$ and $\xi_\mu \otimes F_i\eta$. This space has a unique, up to a scalar, vector killed by E_i, namely,

$$[\mu(i)]_{q_i}\xi_\mu \otimes F_i\xi_\eta - q_i^{\mu(i)}[\eta(i)]_{q_i}F_i\xi_\mu \otimes \xi_\eta.$$

In other words, the isotypic component of $V_\mu \otimes V_\eta$ with highest weight $\mu + \eta - \alpha_i$ is the image of the morphism

$$\tau_{i;\mu,\eta}\colon V_{\mu+\eta-\alpha_i} \to V_\mu \otimes V_\eta, \quad \xi_{\mu+\eta-\alpha_i} \mapsto [\mu(i)]_{q_i}\xi_\mu \otimes F_i\xi_\eta - q_i^{\mu(i)}[\eta(i)]_{q_i}F_i\xi_\mu \otimes \xi_\eta.$$

The action of $\mathcal{E}$ on this image is by a scalar, so there exists $\mathcal{E}_i(\mu, \eta) \in \mathbb{C}^*$ such that

$$\mathcal{E}\tau_{i;\mu,\eta} = \mathcal{E}_i(\mu, \eta)\tau_{i;\mu,\eta}.$$

Lemma 2.3. *Assume the cocycle $\mathcal{E}$ satisfies condition (2.2). Then, for fixed i, the numbers $\mathcal{E}_i(\mu, \eta)$ do not depend on μ and η with $\mu(i), \eta(i) \ge 1$.*

Proof. Consider the module $V_\mu \otimes V_\eta \otimes V_\nu$. The isotypic component corresponding to $\mu + \eta + \nu - \alpha_i$ has multiplicity two, and is spanned by the images of $(\iota \otimes T_{\eta,\nu})\tau_{i;\mu,\eta+\nu}$ and $(\iota \otimes \tau_{i;\eta,\nu})T_{\mu,\eta+\nu-\alpha_i}$, as well as by $(T_{\mu,\eta} \otimes \iota)\tau_{i;\mu+\eta,\nu}$ and $(\tau_{i;\mu,\eta} \otimes \iota)T_{\mu+\eta-\alpha_i,\nu}$. These maps are related by the following identities:

$$[\eta(i)]_{q_i}(T_{\mu,\eta} \otimes \iota)\tau_{i;\mu+\eta,\nu} - [\nu(i)]_{q_i}(\tau_{i;\mu,\eta} \otimes \iota)T_{\mu+\eta-\alpha_i,\nu} = [\mu(i) + \eta(i)]_{q_i}(\iota \otimes \tau_{i;\eta,\nu})T_{\mu,\eta+\nu-\alpha_i}, \tag{2.3}$$

$$[\eta(i) + \nu(i)]_{q_i}(\tau_{i;\mu,\eta} \otimes \iota)T_{\mu+\eta-\alpha_i,\nu} = [\eta(i)]_{q_i}(\iota \otimes T_{\eta,\nu})\tau_{i;\mu,\eta+\nu} - [\mu(i)]_{q_i}(\iota \otimes \tau_{i;\eta,\nu})T_{\mu,\eta+\nu-\alpha_i}. \tag{2.4}$$

These identities are checked by applying both sides to the highest weight vector $\xi_{\mu+\eta+\nu-\alpha_i}$, and using that the T's are module maps, so that for example

$$T_{\mu,\eta}F_i = (F_i \otimes K_i^{-1})T_{\mu,\eta} + (1 \otimes F_i)T_{\mu,\eta}.$$

Applying $(\mathcal{E} \otimes 1)(\hat{\Delta}_q \otimes \iota)(\mathcal{E}) = (1 \otimes \mathcal{E})(\iota \otimes \hat{\Delta}_q)(\mathcal{E})$ to (2.3) and using that $\mathcal{E}T = T$ by (2.2), we get

$$\mathcal{E}_i(\mu + \eta, \nu)[\eta(i)]_{q_i}(T_{\mu,\eta} \otimes \iota)\tau_{i;\mu+\eta,\nu} - \mathcal{E}_i(\mu, \eta)[\nu(i)]_{q_i}(\tau_{i;\mu,\eta} \otimes \iota)T_{\mu+\eta-\alpha_i,\nu}$$
$$= \mathcal{E}_i(\eta, \nu)[\mu(i) + \eta(i)]_{q_i}(\iota \otimes \tau_{i;\eta,\nu})T_{\mu,\eta+\nu-\alpha_i}$$

Since $(T_{\mu,\eta} \otimes \iota)\tau_{i;\mu+\eta,\nu}$ and $(\tau_{i;\mu,\eta} \otimes \iota)T_{\mu+\eta-\alpha_i,\nu}$ are linearly independent, together with (2.3) this implies that

$$\mathcal{E}_i(\mu + \eta, \nu) = \mathcal{E}_i(\mu, \eta) = \mathcal{E}_i(\eta, \nu).$$

Now for arbitrary $\mu, \eta, \tilde{\mu}, \tilde{\eta}$, applying the last identity twice, we get $\mathcal{E}_i(\mu, \eta) = \mathcal{E}_i(\eta, \tilde{\mu}) = \mathcal{E}_i(\tilde{\mu}, \tilde{\eta})$. $\square$

Define a homomorphism $\chi\colon Q \to \mathbb{C}^*$ by letting $\chi(\alpha_i) = \mathcal{E}_i(\mu, \eta)^{-1}$ for $\mu, \eta \in P_+$ with $\mu(i), \eta(i) \ge 1$, $1 \le i \le r$. Extend χ to a homomorphism $P \to \mathbb{C}^*$. The restriction of χ to P_+ defines a central element c of $\mathcal{U}(G_q)$ such that

$$(c \otimes c)\hat{\Delta}_q(c)^{-1}\tau_{i;\mu,\eta} = \chi(\mu)\chi(\eta)\chi(\mu + \eta - \alpha_i)^{-1}\tau_{i;\mu,\eta} = \chi(\alpha_i)\tau_{i;\mu,\eta} = \mathcal{E}_i(\mu, \eta)^{-1}\tau_{i;\mu,\eta}.$$

Thus replacing $\mathcal{E}$ by the cohomologous cocycle $(c \otimes c)\mathcal{E}\hat{\Delta}_q(c)^{-1}$ we get a symmetric invariant 2-cocycle, which we again denote by $\mathcal{E}$, such that

$$\mathcal{E}_i(\mu, \eta) = 1 \quad \text{for all} \quad 1 \le i \le r \quad \text{and} \quad \mu, \eta \in P_+ \quad \text{with} \quad \mu(i), \eta(i) \ge 1. \tag{2.5}$$

Note that condition (2.2) for this new cocycle is still satisfied, since χ is a homomorphism on P_+.

From now on we can and will assume that the symmetric invariant 2-cocycle $\mathcal{E}$ satisfies properties (2.2) and (2.5). We will see later that this already implies that $\mathcal{E} = 1$. But to show this we have to make a rather long detour and first prove that $\mathcal{E}$ is the coboundary of a central element. In the remaining part of the section we will show that for $G = \mathrm{SU}(2)$ this can be avoided.

Recall that for $G = \mathrm{SU}(2)$ the weight lattice P is identified with $\frac{1}{2}\mathbb{Z}$ and the root lattice with $\mathbb{Z}$. For $s \in \frac{1}{2}\mathbb{Z}_+$, we have $V_{1/2} \otimes V_s \cong V_{s+1/2} \oplus V_{s-1/2}$. Therefore conditions (2.2) and (2.5) imply that $\mathcal{E}$ acts trivially on $V_{1/2} \otimes V_s$.

Now for $s, t \geq 1/2$ consider the morphism $T_{1/2,s} \otimes \iota \colon V_{s+1/2} \otimes V_t \to V_{1/2} \otimes V_s \otimes V_t$ and compute

$$
\begin{aligned}
(T_{1/2,s} \otimes \iota)\mathcal{E} &= (\hat{\Delta}_q \otimes \iota)(\mathcal{E})(T_{1/2,s} \otimes \iota) \\
&= (1 \otimes \mathcal{E})(\iota \otimes \hat{\Delta}_q)(\mathcal{E})(\mathcal{E}^{-1} \otimes 1)(T_{1/2,s} \otimes \iota) \\
&= (1 \otimes \mathcal{E})(T_{1/2,s} \otimes \iota),
\end{aligned}
$$

since $\mathcal{E}$ acts trivially on $V_{1/2} \otimes V$ for any V. It follows that if $\mathcal{E}$ acts trivially on $V_s \otimes V_t$, it acts trivially on $V_{s+1/2} \otimes V_t$. Therefore an induction argument shows that $\mathcal{E}$ acts trivially on $V_s \otimes V_t$ for all s and t, so $\mathcal{E} = 1$.

For general G one can similarly show that it suffices to check that $\mathcal{E}$ acts trivially on $V_{\omega_i} \otimes V_\mu$, but we don't know whether it is possible to check the latter property directly using conditions (2.2) and (2.5).

3. Comonoid representing the canonical fiber functor

Consider the automorphism θ of $U_q\mathfrak{g}$ defined by

$$
\theta(E_i) = F_i, \quad \theta(F_i) = E_i, \quad \theta(K_i) = K_i^{-1}.
$$

Observe that $\hat{\Delta}_q\theta = (\theta \otimes \theta)\hat{\Delta}_q^{op}$. For every $U_q\mathfrak{g}$-module V define a module $\bar{V}$ which coincides with V as a vector space, but the action of $U_q\mathfrak{g}$ is given by

$$
X\bar{v} = \overline{\theta(X)v},
$$

where $\bar{v}$ means the vector $v \in V$ considered as an element of $\bar{V}$. Notice that $\bar{\xi}_\mu$ is a lowest weight vector of weight $-\mu$.

Denote by $\bar{\mu}$ the weight $-w_0\mu$, where w_0 is the longest element in the Weyl group. The involution $\lambda \mapsto \bar{\lambda}$ on P defines an involution on the index set $\{1, \ldots, r\}$, so that $\bar{\alpha}_i = \alpha_{\bar{i}}$ and $\bar{\omega}_i = \omega_{\bar{i}}$. It is known that the lowest weight of V_μ is $-\bar{\mu}$. It follows that $\bar{V}_\mu \cong V_{\bar{\mu}}$.

For each $\mu \in P_+$ there exists a unique up to a scalar morphism $\bar{V}_\mu \otimes V_\mu \to V_0 = \mathbb{C}$. Namely, define

$$
S_\mu \colon \bar{V}_\mu \otimes V_\mu \to \mathbb{C}, \quad \bar{\xi}_\mu \otimes \xi_\mu \mapsto 1,
$$

see e.g. [10, Proposition 25.1.4].

For $\mu, \eta \in P_+$, define a morphism

$$
\bar{T}_{\mu,\eta} \colon \bar{V}_{\mu+\eta} \to \bar{V}_\mu \otimes \bar{V}_\eta \quad \text{by} \quad \bar{\xi}_{\mu+\eta} \mapsto \bar{\xi}_\mu \otimes \bar{\xi}_\eta.
$$

For $\lambda \in P$ and $\mu, \eta \in P_+$ such that $\lambda + \mu \in P_+$ consider the morphism

$$
\mathrm{tr}_{\mu,\lambda+\mu}^\eta \colon \bar{V}_{\mu+\eta} \otimes V_{\lambda+\mu+\eta} \to \bar{V}_\mu \otimes V_{\lambda+\mu}, \quad \bar{\xi}_{\mu+\eta} \otimes \xi_{\lambda+\mu+\eta} \mapsto \bar{\xi}_\mu \otimes \xi_{\lambda+\mu}.
$$

Since $\bar{\xi}_{\mu+\eta} \otimes \xi_{\lambda+\mu+\eta}$ is a cyclic vector, its image completely determines the morphism, if it exists. To show existence, rewrite this morphism as the composition

$$
\bar{V}_{\mu+\eta} \otimes V_{\lambda+\mu+\eta} \xrightarrow{\bar{T}_{\mu,\eta} \otimes T_{\eta,\lambda+\mu}} \bar{V}_\mu \otimes \bar{V}_\eta \otimes V_\eta \otimes V_{\lambda+\mu} \xrightarrow{\iota \otimes S_\eta \otimes \iota} \bar{V}_\mu \otimes V_{\lambda+\mu}.
$$

Using the morphisms tr define the inverse limit $U_q\mathfrak{g}$-module

$$M_\lambda = \varprojlim_\mu \bar{V}_\mu \otimes V_{\lambda+\mu}.$$

We consider M_λ as a topological $U_q\mathfrak{g}$-module with a base of neighborhoods of zero formed by the kernels of the canonical morphisms $M_\lambda \to \bar{V}_\mu \otimes V_{\lambda+\mu}$. Observe that $\mathrm{tr}^\eta_{\mu,\lambda+\mu}$ is surjective since its image contains the cyclic vector $\bar{\xi}_\mu \otimes \xi_{\lambda+\mu} \in \bar{V}_\mu \otimes V_{\lambda+\mu}$. It follows that the morphisms $M_\lambda \to \bar{V}_\mu \otimes V_{\lambda+\mu}$ are surjective. Hence, if V is a $U_q\mathfrak{g}$-module with discrete topology, then any continuous morphism $M_\lambda \to V$ factors through $\bar{V}_\mu \otimes V_{\lambda+\mu}$ for some μ, so that the space $\mathrm{Hom}_\mathfrak{g}(M_\lambda, V)$ of such morphisms is the inductive limit of $\mathrm{Hom}_{U_q\mathfrak{g}}(\bar{V}_\mu \otimes V_{\lambda+\mu}, V)$.

Recall, see e.g. [10, Proposition 23.3.10], that if V is an admissible finite dimensional $U_q\mathfrak{g}$-module and λ an integral weight then the map

$$\mathrm{Hom}_{U_q\mathfrak{g}}(\bar{V}_\mu \otimes V_{\lambda+\mu}, V) \to V(\lambda), \quad f \mapsto f(\bar{\xi}_\mu \otimes \xi_{\lambda+\mu}), \tag{3.1}$$

is an isomorphism for sufficiently large dominant integral weights μ. In particular, for any $V \in \mathcal{C}(\mathfrak{g},\hbar)$ the maps (3.1) induce a linear isomorphism

$$\mathrm{Hom}_{U_q\mathfrak{g}}(M_\lambda, V) \to V(\lambda).$$

Therefore the topological $U_q\mathfrak{g}$-module $M = \oplus_{\lambda \in P} M_\lambda$ represents the forgetful functor $\mathcal{C}(\mathfrak{g},\hbar) \to \mathcal{V}ec$. Let

$$\eta_V \colon \mathrm{Hom}_{U_q\mathfrak{g}}(M,V) \to V$$

be the canonical isomorphism.

Our next goal is to define a comonoid structure on M. Define

$$M_{\lambda_1}\hat{\otimes}M_{\lambda_2} = \varprojlim_{\mu_1,\mu_2} (\bar{V}_{\mu_1} \otimes V_{\lambda_1+\mu_1}) \otimes (\bar{V}_{\mu_2} \otimes V_{\lambda_2+\mu_2})$$

and then

$$M\hat{\otimes}M = \prod_{\lambda_1,\lambda_2 \in P} M_{\lambda_1}\hat{\otimes}M_{\lambda_2}.$$

Higher tensor powers of M are defined similarly. We want to define a morphism

$$\delta \colon M \to M\hat{\otimes}M.$$

The restriction of δ to M_λ composed with the projection $M\hat{\otimes}M \to M_{\lambda_1}\hat{\otimes}M_{\lambda_2}$ will be nonzero only if $\lambda = \lambda_1 + \lambda_2$, so δ is determined by maps

$$\delta_{\lambda_1,\lambda_2} \colon M_{\lambda_1+\lambda_2} \to M_{\lambda_1}\hat{\otimes}M_{\lambda_2}.$$

We define these morphisms using the morphisms

$$m_{\mu,\eta,\lambda_1,\lambda_2} \colon \bar{V}_{\mu+\eta} \otimes V_{\lambda_1+\lambda_2+\mu+\eta} \to \bar{V}_\mu \otimes V_{\lambda_1+\mu} \otimes \bar{V}_\eta \otimes V_{\lambda_2+\eta}$$

mapping $\bar{\xi}_{\mu+\eta} \otimes \xi_{\lambda_1+\lambda_2+\mu+\eta}$ onto $\bar{\xi}_\mu \otimes \xi_{\lambda_1+\mu} \otimes \bar{\xi}_\eta \otimes \xi_{\lambda_2+\eta}$. Since $\bar{\xi}_{\mu+\eta} \otimes \xi_{\lambda_1+\lambda_2+\mu+\eta}$ is a cyclic vector, such a morphism is unique if it exists, and to show its existence we rewrite it, using property (2.1) of the R-matrix, as the composition

$$\bar{V}_{\mu+\eta} \otimes V_{\lambda_1+\lambda_2+\mu+\eta} \xrightarrow{\bar{T}_{\mu,\eta} \otimes T_{\lambda_1+\mu,\lambda_2+\eta}} \bar{V}_\mu \otimes \bar{V}_\eta \otimes V_{\lambda_1+\mu} \otimes V_{\lambda_2+\eta}$$

$$\xrightarrow{q^{(\lambda_1+\mu,\eta)}(\iota\otimes\sigma\otimes\iota)} \bar{V}_\mu \otimes V_{\lambda_1+\mu} \otimes \bar{V}_\eta \otimes V_{\lambda_2+\eta}.$$

The morphisms m are consistent with the morphisms tr defining the inverse limits, that is,

$$(\mathrm{tr}^\nu_{\mu,\lambda_1+\mu} \otimes \mathrm{tr}^\omega_{\eta,\lambda_2+\eta})m_{\mu+\nu,\eta+\omega,\lambda_1,\lambda_2} = m_{\mu,\eta,\lambda_1,\lambda_2}\,\mathrm{tr}^{\nu+\omega}_{\mu+\eta,\lambda_1+\lambda_2+\mu+\eta}.$$

Hence they define morphisms $\delta_{\lambda_1,\lambda_2} \colon M_{\lambda_1+\lambda_2} \to M_{\lambda_1}\hat{\otimes}M_{\lambda_2}$.

Using the morphisms $\delta_{\lambda_1,\lambda_2}$ we can in an obvious way define morphisms

$$(\delta \otimes \iota)\delta, (\iota \otimes \delta)\delta \colon M \to M\hat{\otimes}M\hat{\otimes}M.$$

We also introduce a morphism $\varepsilon\colon M \to \mathbb{C}$ by requiring it to be nonzero only on M_0, where we set it to be the canonical morphism $M_0 \to \bar{V}_0 \otimes V_0 = \mathbb{C}$, so that $\varepsilon\colon M_0 \to \mathbb{C}$ is determined by the morphisms

$$\mathrm{tr}^{\mu}_{0,0} = S_\mu\colon \bar{V}_\mu \otimes V_\mu \to \mathbb{C}.$$

Proposition 3.1. *The triple (M, δ, ε) is a comonoid representing the canonical fiber functor $\mathcal{C}(\mathfrak{g}, \hbar) \to \mathcal{V}ec$, that is,*

$$(\delta \otimes \iota)\delta = (\iota \otimes \delta)\delta, \quad (\varepsilon \otimes \iota)\delta = \iota = (\iota \otimes \varepsilon)\delta,$$

and for all $U, V \in \mathcal{C}(\mathfrak{g}, \hbar)$ the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Hom}_{U_q\mathfrak{g}}(M,U) \otimes \mathrm{Hom}_{U_q\mathfrak{g}}(M,V) & \xrightarrow{\eta_U \otimes \eta_V} & U \otimes V \\
\downarrow & & \| \\
\mathrm{Hom}_{U_q\mathfrak{g}}(M, U \otimes V) & \xrightarrow{\eta_{U \otimes V}} & U \otimes V
\end{array}
\quad,
$$

where the left vertical arrow is given by $f \otimes g \mapsto (f \otimes g)\delta$.

Proof. For $\lambda_1, \lambda_2, \lambda_3 \in P$ we have to check that

$$(\delta_{\lambda_1,\lambda_2} \otimes \iota)\delta_{\lambda_1+\lambda_2,\lambda_3} = (\iota \otimes \delta_{\lambda_2,\lambda_3})\delta_{\lambda_1,\lambda_2+\lambda_3}.$$

This reduces to showing that

$$(m_{\mu_1,\mu_2,\lambda_1,\lambda_2} \otimes \iota \otimes \iota)m_{\mu_1+\mu_2,\mu_3,\lambda_1+\lambda_2,\lambda_3} = (\iota \otimes \iota \otimes m_{\mu_2,\mu_3,\lambda_2,\lambda_3})m_{\mu_1,\mu_2+\mu_3,\lambda_1,\lambda_2+\lambda_3},$$

which follows immediately by definition.

Next we have to check that on M_λ we have $(\varepsilon \otimes \iota)\delta_{0,\lambda} = \iota = (\iota \otimes \varepsilon)\delta_{\lambda,0}$. This is again straightforward.

Finally, to check commutativity of the diagram recall that the isomorphism

$$\mathrm{Hom}_{U_q\mathfrak{g}}(M_{\lambda_1}, V_\mu) \to V_\mu(\lambda_1)$$

comes from the homomorphisms $\mathrm{Hom}_{U_q\mathfrak{g}}(\bar{V}_\nu \otimes V_{\lambda_1+\nu}, V_\mu) \to V_\mu(\lambda_1)$ given by $f \mapsto f(\bar{\xi}_\nu \otimes \xi_{\lambda_1+\nu})$. It follows that it suffices to check that

$$m_{\nu,\omega,\lambda_1,\lambda_2}(\bar{\xi}_{\nu+\omega} \otimes \xi_{\lambda_1+\lambda_2+\nu+\omega}) = \bar{\xi}_\nu \otimes \xi_{\lambda_1+\nu} \otimes \bar{\xi}_\omega \otimes \xi_{\lambda_2+\omega},$$

but this is exactly the definition of m. $\qquad\qquad\square$

The algebra $U_q\mathfrak{g}$ acts by endomorphisms of the forgetful functor $\mathcal{C}(\mathfrak{g}, \hbar) \to \mathcal{V}ec$. Our next goal is to show that the generators of this action lift to endomorphisms of M.

Recall that in the previous section we defined morphisms

$$\tau_{i;\eta,\mu}\colon V_{\eta+\mu-\alpha_i} \to V_\eta \otimes V_\mu, \quad \xi_{\eta+\mu-\alpha_i} \mapsto [\eta(i)]_{q_i}\xi_\eta \otimes F_i\xi_\mu - q_i^{\eta(i)}[\mu(i)]_{q_i} F_i\xi_\eta \otimes \xi_\mu.$$

Similarly, define morphisms

$$\bar{\tau}_{i;\mu,\eta}\colon \bar{V}_{\mu+\eta-\alpha_i} \to \bar{V}_\mu \otimes \bar{V}_\eta, \quad \bar{\xi}_{\mu+\eta-\alpha_i} \mapsto [\eta(i)]_{q_i} E_i\bar{\xi}_\mu \otimes \bar{\xi}_\eta - q_i^{\eta(i)}[\mu(i)]_{q_i}\bar{\xi}_\mu \otimes E_i\bar{\xi}_\eta.$$

Equivalently, $\bar{\tau}_{i;\mu,\eta} = \Sigma\tau_{i;\eta,\mu}$.

Consider the morphism

$$\Psi^{\eta}_{i;\mu,\lambda+\alpha_i+\mu}\colon \bar{V}_{\mu+\eta} \otimes V_{\lambda+\mu+\eta} \to \bar{V}_\mu \otimes V_{\lambda+\alpha_i+\mu}, \quad \bar{\xi}_{\mu+\eta} \otimes \xi_{\lambda+\mu+\eta} \mapsto \bar{\xi}_\mu \otimes F_i\xi_{\lambda+\alpha_i+\mu}.$$

To see that it is well-defined, rewrite it as the composition

$$\bar{V}_{\mu+\eta} \otimes V_{\lambda+\mu+\eta} \xrightarrow{[\eta(i)]_{q_i}^{-1}\bar{T}_{\mu,\eta} \otimes \tau_{i;\eta,\lambda+\alpha_i+\mu}} \bar{V}_\mu \otimes \bar{V}_\eta \otimes V_\eta \otimes V_{\lambda+\alpha_i+\mu} \xrightarrow{\iota \otimes S_\eta \otimes \iota} \bar{V}_\mu \otimes V_{\lambda+\alpha_i+\mu}.$$

Since $\bar{\xi}_\mu \otimes F_i\xi_{\lambda+\alpha_i+\mu} = \hat{\Delta}_q(F_i)(\bar{\xi}_\mu \otimes \xi_{\lambda+\alpha_i+\mu})$, the morphisms Ψ are consistent with tr and hence define a morphism $\bar{F}_i\colon M_\lambda \to M_{\lambda+\alpha_i}$.

Similarly, consider the morphism

$$\Phi^\eta_{i;\mu+\alpha_i,\lambda+\mu}\colon \bar{V}_{\mu+\eta}\otimes V_{\lambda+\mu+\eta}\to \bar{V}_{\mu+\alpha_i}\otimes V_{\lambda+\mu},\ \ \bar{\xi}_{\mu+\eta}\otimes\xi_{\lambda+\mu+\eta}\mapsto E_i\bar{\xi}_{\mu+\alpha_i}\otimes\xi_{\lambda+\mu},$$

which can be equivalently written as the composition

$$\bar{V}_{\mu+\eta}\otimes V_{\lambda+\mu+\eta}\xrightarrow{[\eta(i)]_{q_i}^{-1}\bar{\tau}_{i;\mu+\alpha_i,\eta}\otimes T_{\eta,\lambda+\mu}}\bar{V}_{\mu+\alpha_i}\otimes\bar{V}_\eta\otimes V_\eta\otimes V_{\lambda+\mu}\xrightarrow{\iota\otimes S_\eta\otimes\iota}\bar{V}_{\mu+\alpha_i}\otimes V_{\lambda+\mu}.$$

Again, using that $E_i\bar{\xi}_{\mu+\alpha_i}\otimes\xi_{\lambda+\mu}=\hat{\Delta}_q(E_i)(\bar{\xi}_{\mu+\alpha_i}\otimes\xi_{\lambda+\mu})$, we see that the morphisms Φ are consistent with tr and hence define a morphism $\tilde{E}_i\colon M_\lambda\to M_{\lambda-\alpha_i}$.

Define also a morphism $\tilde{K}_i\colon M\to M$ by $\tilde{K}_i|_{M_\lambda}=q_i^{\lambda(i)}$.

Proposition 3.2. *For all $1\le i\le r$ and $V\in\mathcal{C}(\mathfrak{g},\hbar)$ the following diagrams commute:*

$$
\begin{array}{ccc}
\mathrm{Hom}_{U_q\mathfrak{g}}(M,V)\xrightarrow{\eta_V}V & \mathrm{Hom}_{U_q\mathfrak{g}}(M,V)\xrightarrow{\eta_V}V & \mathrm{Hom}_{U_q\mathfrak{g}}(M,V)\xrightarrow{\eta_V}V\quad.\\[2pt]
\Big\downarrow{\scriptstyle\circ\tilde{E}_i}\qquad\Big\downarrow{\scriptstyle E_i} & \Big\downarrow{\scriptstyle\circ\tilde{F}_i}\qquad\Big\downarrow{\scriptstyle F_i} & \Big\downarrow{\scriptstyle\circ\tilde{K}_i}\qquad\Big\downarrow{\scriptstyle K_i}\\[2pt]
\mathrm{Hom}_{U_q\mathfrak{g}}(M,V)\xrightarrow{\eta_V}V & \mathrm{Hom}_{U_q\mathfrak{g}}(M,V)\xrightarrow{\eta_V}V & \mathrm{Hom}_{U_q\mathfrak{g}}(M,V)\xrightarrow{\eta_V}V
\end{array}
$$

Proof. To show commutativity of the first diagram it suffices to check that if

$$f\in\mathrm{Hom}_{U_q\mathfrak{g}}(\bar{V}_{\mu+\alpha_i}\otimes V_{\lambda+\mu},V)$$

then $E_if(\bar{\xi}_{\mu+\alpha_i}\otimes\xi_{\lambda+\mu})=f\Phi^\eta_{i;\mu+\alpha_i,\lambda+\mu}(\bar{\xi}_{\mu+\eta}\otimes\xi_{\lambda+\mu+\eta})$. Since

$$\Phi^\eta_{i;\mu+\alpha_i,\lambda+\mu}(\bar{\xi}_{\mu+\eta}\otimes\xi_{\lambda+\mu+\eta})=\hat{\Delta}_q(E_i)(\bar{\xi}_{\mu+\alpha_i}\otimes\xi_{\lambda+\mu}),$$

this is indeed true. The second diagram commutes for similar reasons, while commutativity of the third diagram is obvious. $\qquad\square$

4. Proof of the main theorem

We now return to the proof of Theorem 2.1. So let $\mathcal{E}\in\mathcal{U}(G_q\times G_q)$ be a symmetric invariant 2-cocycle satisfying properties (2.2) and (2.5). Recall that the latter properties mean that

$$\mathcal{E}T_{\mu,\eta}=T_{\mu,\eta}\ \ \text{and}\ \ \mathcal{E}\tau_{i;\mu,\eta}=\tau_{i;\mu,\eta}.$$

In the previous section we also introduced the maps $\bar{T}_{\mu,\eta}$ and $\bar{\tau}_{i;\mu,\eta}$. The first is an isomorphism of $\bar{V}_{\mu+\eta}$ onto the isotypic component of $\bar{V}_\mu\otimes\bar{V}_\eta$ with lowest weight $-\mu-\eta$, that is, with highest weight $\bar{\mu}+\bar{\eta}$. The second is an isomorphism of $\bar{V}_{\mu+\eta-\alpha_i}$ onto the isotypic component with lowest weight $-\mu-\eta+\alpha_i$, hence with highest weight $\bar{\mu}+\bar{\eta}-\bar{\alpha}_i$. Therefore if we fix isomorphisms $\bar{V}_\nu\cong V_{\bar{\nu}}$, then $\bar{T}_{\mu,\eta}$ and $\bar{\tau}_{i;\mu,\eta}$ coincide with $T_{\bar{\mu},\bar{\eta}}$ and $\tau_{i;\bar{\mu},\bar{\eta}}$ up to scalar factors. Hence properties (2.2) and (2.5) also imply that

$$\mathcal{E}\bar{T}_{\mu,\eta}=\bar{T}_{\mu,\eta}\ \ \text{and}\ \ \mathcal{E}\bar{\tau}_{i;\mu,\eta}=\bar{\tau}_{i;\mu,\eta}.$$

Since $\mathcal{E}$ is invertible, the morphism $S_\mu\mathcal{E}\colon\bar{V}_\mu\otimes V_\mu\to\mathbb{C}$ is nonzero, hence it is a nonzero multiple of S_μ, so $S_\mu\mathcal{E}=\chi(\mu)S_\mu$ for some $\chi(\mu)\in\mathbb{C}^*$. Explicitly, $\chi(\mu)=S_\mu\mathcal{E}(\bar{\xi}_\mu\otimes\xi_\mu)$.

Lemma 4.1. *For all $\mu,\eta\in P_+$ and $\lambda\in P$ such that $\lambda+\mu\in P_+$ we have $\mathrm{tr}^\eta_{\mu,\lambda+\mu}\mathcal{E}=\chi(\eta)\mathcal{E}\,\mathrm{tr}^\eta_{\mu,\lambda+\mu}$.*

Proof. Applying $\iota\otimes\iota\otimes\hat{\Delta}_q$ to the cocycle identity

$$(\mathcal{E}\otimes 1)(\hat{\Delta}_q\otimes\iota)(\mathcal{E})=(1\otimes\mathcal{E})(\iota\otimes\hat{\Delta}_q)(\mathcal{E}),$$

we get

$$(\mathcal{E}\otimes 1\otimes 1)(\hat{\Delta}_q\otimes\hat{\Delta}_q)(\mathcal{E})=(1\otimes(\iota\otimes\hat{\Delta}_q)(\mathcal{E}))(\iota\otimes\hat{\Delta}_q^{(2)})(\mathcal{E}),$$

where $\hat{\Delta}_q^{(2)} = (\iota \otimes \hat{\Delta}_q)\hat{\Delta}_q$. Replacing $(\iota \otimes \hat{\Delta}_q)(\mathcal{E})$ by $(1 \otimes \mathcal{E}^{-1})(\mathcal{E} \otimes 1)(\hat{\Delta}_q \otimes \iota)(\mathcal{E})$ on the right hand side, we then get

$$(\mathcal{E} \otimes \mathcal{E})(\hat{\Delta}_q \otimes \hat{\Delta}_q)(\mathcal{E}) = (1 \otimes \mathcal{E} \otimes 1)(1 \otimes (\hat{\Delta}_q \otimes \iota)(\mathcal{E}))(\iota \otimes \hat{\Delta}_q^{(2)})(\mathcal{E}),$$

which can also be written as

$$(\hat{\Delta}_q \otimes \hat{\Delta}_q)(\mathcal{E}) = (1 \otimes \mathcal{E} \otimes 1)(1 \otimes (\hat{\Delta}_q \otimes \iota)(\mathcal{E}))(\iota \otimes \hat{\Delta}_q^{(2)})(\mathcal{E})(\mathcal{E}^{-1} \otimes \mathcal{E}^{-1}), \qquad (4.1)$$

since $\mathcal{E}$ commutes with the image of $\hat{\Delta}_q$ by G_q-invariance.

We then compute

$$\begin{aligned}
\mathrm{tr}_{\mu,\lambda+\mu}^{\eta} \mathcal{E} &= (\iota \otimes S_\eta \otimes \iota)(\bar{T}_{\mu,\eta} \otimes T_{\eta,\lambda+\mu})\mathcal{E} \\
&= (\iota \otimes S_\eta \otimes \iota)(\hat{\Delta}_q \otimes \hat{\Delta}_q)(\mathcal{E})(\bar{T}_{\mu,\eta} \otimes T_{\eta,\lambda+\mu}) \\
&= (\iota \otimes S_\eta \otimes \iota)(1 \otimes \mathcal{E} \otimes 1)(1 \otimes (\hat{\Delta}_q \otimes \iota)(\mathcal{E}))(\iota \otimes \hat{\Delta}_q^{(2)})(\mathcal{E})(\mathcal{E}^{-1} \otimes \mathcal{E}^{-1})(\bar{T}_{\mu,\eta} \otimes T_{\eta,\lambda+\mu}) \\
&= (\iota \otimes S_\eta \otimes \iota)(1 \otimes \mathcal{E} \otimes 1)(1 \otimes (\hat{\Delta}_q \otimes \iota)(\mathcal{E}))(\iota \otimes \hat{\Delta}_q^{(2)})(\mathcal{E})(\bar{T}_{\mu,\eta} \otimes T_{\eta,\lambda+\mu}) \\
&\qquad \text{(by condition (2.2))} \\
&= \chi(\eta)(\iota \otimes S_\eta \otimes \iota)(\iota \otimes \hat{\Delta}_q^{(2)})(\mathcal{E})(\bar{T}_{\mu,\eta} \otimes T_{\eta,\lambda+\mu}) \\
&\qquad \text{(since } S_\eta \hat{\Delta}_q(\omega) = \hat{\varepsilon}_q(\omega)S_\eta \text{ and } (\hat{\varepsilon}_q \otimes \iota)(\mathcal{E}) = 1 \text{ by (2.2))} \\
&= \chi(\eta)\mathcal{E}(\iota \otimes S_\eta \otimes \iota)(\bar{T}_{\mu,\eta} \otimes T_{\eta,\lambda+\mu}) \\
&\qquad \text{(since } S_\eta \hat{\Delta}_q(\omega) = \hat{\varepsilon}_q(\omega)S_\eta \text{ and } (\hat{\varepsilon}_q \otimes \iota)\hat{\Delta}_q = \iota) \\
&= \chi(\eta)\mathcal{E}\, \mathrm{tr}_{\mu,\lambda+\mu}^{\eta}.
\end{aligned}$$

$\qquad \square$

In particular, using that $S_{\mu+\eta} = \mathrm{tr}_{0,0}^{\mu+\eta} = \mathrm{tr}_{0,0}^{\mu}\, \mathrm{tr}_{\mu,\mu}^{\eta}$ we get

$$\chi(\mu + \eta)S_{\mu+\eta} = S_{\mu+\eta}\mathcal{E} = S_\mu\, \mathrm{tr}_{\mu,\mu}^{\eta}\, \mathcal{E} = \chi(\eta)S_\mu\mathcal{E}\, \mathrm{tr}_{\mu,\mu}^{\eta} = \chi(\eta)\chi(\mu)S_\mu\, \mathrm{tr}_{\mu,\mu}^{\eta} = \chi(\eta)\chi(\mu)S_{\mu+\eta}.$$

Thus the map $\chi \colon P_+ \to \mathbb{C}^*$ is a homomorphism, hence it extends to a homomorphism $P \to \mathbb{C}^*$, which we continue to denote by χ. This together with the above lemma implies that the morphisms

$$\chi(\mu)^{-1}\mathcal{E} \colon \bar{V}_\mu \otimes V_{\lambda+\mu} \to \bar{V}_\mu \otimes V_{\lambda+\mu}$$

are consistent with tr, hence define a morphism $\mathcal{E}_0 \colon M_\lambda \to M_\lambda$. Note that $\mathcal{E}_0$ is invertible since $\mathcal{E}$ is.

Lemma 4.2. *For all $1 \le i \le r$ we have*

$$\tilde{E}_i \mathcal{E}_0 = \chi(\alpha_i)\mathcal{E}_0\tilde{E}_i, \quad \tilde{F}_i\mathcal{E}_0 = \mathcal{E}_0\tilde{F}_i \quad and \quad \tilde{K}_i\mathcal{E}_0 = \mathcal{E}_0\tilde{K}_i.$$

Proof. Recall that $\tilde{E}_i$ is defined using the morphisms $\Phi_{i;\mu+\alpha_i,\lambda+\mu}^{\eta}$ given by the composition

$$\bar{V}_{\mu+\eta} \otimes V_{\lambda+\mu+\eta} \xrightarrow{[\eta(i)]_{q_i}^{-1}\bar{\tau}_{i;\mu+\alpha_i,\eta} \otimes T_{\eta,\lambda+\mu}} \bar{V}_{\mu+\alpha_i} \otimes \bar{V}_\eta \otimes V_\eta \otimes V_{\lambda+\mu} \xrightarrow{\iota \otimes S_\eta \otimes \iota} \bar{V}_{\mu+\alpha_i} \otimes V_{\lambda+\mu}.$$

The same proof as that in Lemma 4.1 shows that

$$\Phi_{i;\mu+\alpha_i,\lambda+\mu}^{\eta}\mathcal{E} = \chi(\eta)\mathcal{E}\Phi_{i;\mu+\alpha_i,\lambda+\mu}^{\eta}.$$

The only difference is that $\bar{T}_{\mu,\eta}$ in that lemma gets replaced by $\bar{\tau}_{i;\mu+\alpha_i,\eta}$ and then instead of condition (2.2) one uses condition (2.5). Dividing both sides of the above identity by $\chi(\mu + \eta)$, we get $\tilde{E}_i\mathcal{E}_0 = \chi(\alpha_i)\mathcal{E}_0\tilde{E}_i$.

Similarly, $\tilde{F}_i$ is defined using the morphisms $\Psi_{i;\mu,\lambda+\alpha_i+\mu}^{\eta}$ given by the composition

$$\bar{V}_{\mu+\eta} \otimes V_{\lambda+\mu+\eta} \xrightarrow{[\eta(i)]_{q_i}^{-1}\bar{T}_{\mu,\eta} \otimes \tau_{i;\eta,\lambda+\alpha_i+\mu}} \bar{V}_\mu \otimes \bar{V}_\eta \otimes V_\eta \otimes V_{\lambda+\alpha_i+\mu} \xrightarrow{\iota \otimes S_\eta \otimes \iota} \bar{V}_\mu \otimes V_{\lambda+\alpha_i+\mu}.$$

It follows that

$$\Psi_{i;\mu,\lambda+\alpha_i+\mu}^{\eta}\mathcal{E} = \chi(\eta)\mathcal{E}\Psi_{i;\mu,\lambda+\alpha_i+\mu}^{\eta},$$

and dividing both sides by $\chi(\mu + \eta)$ we get $\tilde{F}_i \mathcal{E}_0 = \mathcal{E}_0 \tilde{F}_i$.

The commutation with $\tilde{K}_i$ is obvious. $\square$

The morphism $\mathcal{E}_0$ defines an endomorphism of the functor $\mathrm{Hom}_{U_q\mathfrak{g}}(M, \cdot)$. Since this functor is isomorphic to the forgetful functor and the algebra of endomorphisms of the forgetful functor is $\mathcal{U}(G_q)$, the morphism $\mathcal{E}_0$ defines an invertible element $c \in \mathcal{U}(G_q)$ such that for any $V \in \mathcal{C}(\mathfrak{g}, \hbar)$ the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Hom}_{U_q\mathfrak{g}}(M, V) & \xrightarrow{\eta_V} & V \\
{\scriptstyle \circ \mathcal{E}_0} \downarrow & & \downarrow {\scriptstyle c} \\
\mathrm{Hom}_{U_q\mathfrak{g}}(M, V) & \xrightarrow{\eta_V} & V
\end{array}
\qquad (4.2)
$$

Proposition 3.2 and Lemma 4.2 imply that

$$
cE_i = \chi(\alpha_i)E_ic, \quad cF_i = F_ic \quad \text{and} \quad cK_i = K_ic.
$$

Since $E_iF_i - F_iE_i$ coincides with $K_i - K_i^{-1}$ up to a scalar factor, this is possible only if $\chi(\alpha_i) = 1$. Therefore c belongs to the center of $\mathcal{U}(G_q)$.

Lemma 4.3. *We have* $\delta\mathcal{E}_0 = \mathcal{E}(\mathcal{E}_0 \otimes \mathcal{E}_0)\delta$.

Proof. Recall that δ is defined using the morphisms

$$
m_{\mu,\eta,\lambda_1,\lambda_2} : \bar{V}_{\mu+\eta} \otimes V_{\lambda_1+\lambda_2+\mu+\eta} \to \bar{V}_\mu \otimes V_{\lambda_1+\mu} \otimes \bar{V}_\eta \otimes V_{\lambda_2+\eta}
$$

given by $m_{\mu,\eta,\lambda_1,\lambda_2} = q^{(\lambda_1+\mu,\eta)}(\iota \otimes \sigma \otimes \iota)(\bar{T}_{\mu,\eta} \otimes T_{\lambda_1+\mu,\lambda_2+\eta})$. The same computation as that in Lemma 4.1 shows that

$$
\begin{aligned}
m_{\mu,\eta,\lambda_1,\lambda_2}\mathcal{E} &= q^{(\lambda_1+\mu,\eta)}(\iota \otimes \sigma \otimes \iota)(1 \otimes \mathcal{E} \otimes 1)(1 \otimes (\hat{\Delta}_q \otimes \iota)(\mathcal{E}))(\iota \otimes \hat{\Delta}_q^{(2)})(\mathcal{E})(\bar{T}_{\mu,\eta} \otimes T_{\lambda_1+\mu,\lambda_2+\eta}) \\
&= q^{(\lambda_1+\mu,\eta)}(1 \otimes \mathcal{E} \otimes 1)(1 \otimes (\hat{\Delta}_q \otimes \iota)(\mathcal{E}))(\iota \otimes \hat{\Delta}_q^{(2)})(\mathcal{E})(\iota \otimes \sigma \otimes \iota)(\bar{T}_{\mu,\eta} \otimes T_{\lambda_1+\mu,\lambda_2+\eta}) \\
&\qquad \text{(since } \sigma\mathcal{E} = \mathcal{E}\sigma \text{ by the assumption that } \mathcal{E} \text{ is symmetric)} \\
&= q^{(\lambda_1+\mu,\eta)}(\hat{\Delta}_q \otimes \hat{\Delta}_q)(\mathcal{E})(\mathcal{E} \otimes \mathcal{E})(\iota \otimes \sigma \otimes \iota)(\bar{T}_{\mu,\eta} \otimes T_{\lambda_1+\mu,\lambda_2+\eta}) \\
&\qquad \text{(by (4.1))} \\
&= (\hat{\Delta}_q \otimes \hat{\Delta}_q)(\mathcal{E})(\mathcal{E} \otimes \mathcal{E})m_{\mu,\eta,\lambda_1,\lambda_2},
\end{aligned}
$$

which proves the lemma. $\square$

Proof of Theorem 2.1. For $U, V \in \mathcal{C}(\mathfrak{g}, \hbar)$ and $f \in \mathrm{Hom}_{U_q\mathfrak{g}}(M, U)$, $g \in \mathrm{Hom}_{U_q\mathfrak{g}}(M, V)$ we have

$$
\begin{aligned}
\hat{\Delta}_q(c)(\eta_U(f) \otimes \eta_V(g)) &= \hat{\Delta}_q(c)\eta_{U\otimes V}((f \otimes g)\delta) && \text{(by Proposition 3.1)} \\
&= \eta_{U\otimes V}((f \otimes g)\delta\mathcal{E}_0) && \text{(by (4.2))} \\
&= \eta_{U\otimes V}((f \otimes g)\mathcal{E}(\mathcal{E}_0 \otimes \mathcal{E}_0)\delta) && \text{(by Lemma 4.3)} \\
&= \eta_{U\otimes V}(\mathcal{E}(f \otimes g)(\mathcal{E}_0 \otimes \mathcal{E}_0)\delta) && \\
&= \mathcal{E}\eta_{U\otimes V}((f\mathcal{E}_0 \otimes g\mathcal{E}_0)\delta) && \text{(by naturality of } \eta \text{ and } G_q\text{-invariance of } \mathcal{E}) \\
&= \mathcal{E}(\eta_U(f\mathcal{E}_0) \otimes \eta_V(g\mathcal{E}_0)) && \text{(by Proposition 3.1)} \\
&= \mathcal{E}(c \otimes c)(\eta_U(f) \otimes \eta_V(g)) && \text{(by (4.2))}.
\end{aligned}
$$

It follows that $\hat{\Delta}_q(c) = \mathcal{E}(c \otimes c)$. $\square$

With a bit more work one can show that in fact $\mathcal{E} = 1$.

Corollary 4.4. *If* $\mathcal{E} \in \mathcal{U}(G_q \times G_q)$ *is a symmetric invariant 2-cocycle satisfying properties* (2.2) *and* (2.5), *then* $\mathcal{E} = 1$.

Proof. By Theorem 2.1, $\mathcal{E}$ is the coboundary of a central element, that is, $\mathcal{E} = (c \otimes c)\hat{\Delta}_q(c)^{-1}$. The element c acts on V_μ by a scalar $\chi(\mu)$. Condition (2.2) means then that $\chi \colon P_+ \to \mathbb{C}^*$ is a homomorphism, so χ extends to a homomorphism $P \to \mathbb{C}^*$. Condition (2.5) implies that $\chi(\alpha_i) = 1$ for all i, so χ is trivial on the root lattice Q. In other words, χ is a character of P/Q. But then c is group-like, that is, $\hat{\Delta}_q(c) = c \otimes c$. This is well-known for $q = 1$, since the characters of P/Q are in a one-to-one correspondence with the elements of the center of G, see e.g. [3, Theorem 26.3], and so c belongs to $G \subset \mathcal{U}(G)$ and is therefore group-like. For general q, the canonical identification of the centers of $\mathcal{U}(G_q)$ and $\mathcal{U}(G)$ extends to an isomorphism of algebras, since the dimensions of the irreducible representations with a given highest weight do not depend on q. Since the fusion rules do not depend on q either, there exists $\mathcal{F} \in \mathcal{U}(G \times G)$ such that $\hat{\Delta}_q = \mathcal{F}\hat{\Delta}(\cdot)\mathcal{F}^{-1}$. Then as c is group-like in $(\mathcal{U}(G), \hat{\Delta})$, we have

$$\hat{\Delta}_q(c) = \mathcal{F}\hat{\Delta}(c)\mathcal{F}^{-1} = \mathcal{F}(c \otimes c)\mathcal{F}^{-1} = c \otimes c,$$

and so c is group-like in $(\mathcal{U}(G_q), \hat{\Delta}_q)$ as well. Hence $\mathcal{E} = (c \otimes c)\hat{\Delta}_q(c)^{-1} = 1$. $\qquad\square$

In the above proof we remarked that a central element in $\mathcal{U}(G_q)$ is group-like if it is defined by a character of P/Q. The converse is also true.

Proposition 4.5. *A central element of $\mathcal{U}(G_q)$ is group-like if and only if it is defined by a character of P/Q.*

Proof. We only have to show that if c is central and group-like then it is defined by a character of P/Q. Similarly to the proof of Corollary 4.4, we first conclude that c is group-like in $\mathcal{U}(G)$ as well. Then c is a central element of the complexification $G_\mathbb{C} \subset \mathcal{U}(G)$ of G by Theorem A.1. Hence it belongs to G and is defined by a character of P/Q, see again [3, Theorem 26.3]. $\qquad\square$

Corollary 4.6. *If $\mathcal{E} \in \mathcal{U}(G_q \times G_q)$ is a symmetric invariant 2-cocycle then a central element c such that $\mathcal{E} = (c \otimes c)\hat{\Delta}_q(c)^{-1}$ is defined uniquely up to a character of P/Q.*

5. Uniqueness of the Drinfeld twist

We shall assume throughout this section that $q > 0$. Let $\hbar \in i\mathbb{R}$ be such that $q = e^{\pi i \hbar}$. Denote by $t \in \mathfrak{g} \otimes \mathfrak{g}$ the $\mathfrak{g}$-invariant symmetric element defined by the Killing form normalized so that the induced form on $\mathfrak{h}^*$ satisfies $(\alpha, \alpha) = 2$ for short roots. Let $\Phi_{KZ} = \Phi(\hbar t_{12}, \hbar t_{23}) \in \mathcal{U}(G \times G \times G)$ be the Drinfeld associator defined via monodromy of the KZ-equations, see e.g. [13] for details.

Recall that from the work of Kazhdan and Lusztig [8] one can derive [13] the following analytic version of a famous result of Drinfeld [5, 6].

Theorem 5.1. *For any isomorphism $\varphi \colon \mathcal{U}(G_q) \to \mathcal{U}(G)$ extending the canonical identification of the centers there exists an invertible element $\mathcal{F} \in \mathcal{U}(G \times G)$ such that*

(i) $(\varphi \otimes \varphi)\hat{\Delta}_q = \mathcal{F}\hat{\Delta}\varphi(\cdot)\mathcal{F}^{-1}$;

(ii) $(\hat{\varepsilon} \otimes \iota)(\mathcal{F}) = (\iota \otimes \hat{\varepsilon})(\mathcal{F}) = 1$;

(iii) $(\varphi \otimes \varphi)(\mathcal{R}_\hbar) = \mathcal{F}_{21}q^t\mathcal{F}^{-1}$;

(iv) $\Phi_{KZ} = (\iota \otimes \hat{\Delta})(\mathcal{F}^{-1})(1 \otimes \mathcal{F}^{-1})(\mathcal{F} \otimes 1)(\hat{\Delta} \otimes \iota)(\mathcal{F})$.

In addition, if φ is a $*$-isomorphism then $\mathcal{F}$ can be taken to be unitary.

Any such element $\mathcal{F}$ is called a Drinfeld twist. Our next result asserts that for φ fixed, the Drinfeld twist is unique up to coboundary of a central element. This is an equivalent form of Theorem 2.1.

Theorem 5.2. *Suppose $\mathcal{F}$ and $\mathcal{F}'$ are two Drinfeld twists for the same isomorphism φ. Then there exists a central element c of $\mathcal{U}(G)$ such that $\mathcal{F}' = (c \otimes c)\mathcal{F}\hat{\Delta}(c)^{-1}$. When φ is a $*$-isomorphism and both Drinfeld twists are unitary, then c can also be chosen to be unitary.*

Proof. To simplify the notation we shall omit φ in the computations, so we identify $\mathcal{U}(G_q)$ and $\mathcal{U}(G)$ as algebras. Set $\mathcal{E} = \mathcal{F}'\mathcal{F}^{-1}$. Then

$$(\iota \otimes \hat{\Delta})(\mathcal{F}^{-1})(1 \otimes \mathcal{F}^{-1})(\mathcal{F} \otimes 1)(\hat{\Delta} \otimes \iota)(\mathcal{F}) = (\iota \otimes \hat{\Delta})(\mathcal{F}^{-1}\mathcal{E}^{-1})(1 \otimes \mathcal{F}^{-1}\mathcal{E}^{-1})(\mathcal{E}\mathcal{F} \otimes 1)(\hat{\Delta} \otimes \iota)(\mathcal{E}\mathcal{F}).$$

Multiplying by $(1 \otimes \mathcal{F})(\iota \otimes \hat{\Delta})(\mathcal{F})$ on the left and by $(\hat{\Delta} \otimes \iota)(\mathcal{F}^{-1})(\mathcal{F}^{-1} \otimes 1)$ on the right, and using that $\mathcal{F}\hat{\Delta}(\cdot)\mathcal{F}^{-1} = \hat{\Delta}_q$, we get

$$1 = (\iota \otimes \hat{\Delta}_q)(\mathcal{E}^{-1})(1 \otimes \mathcal{E}^{-1})(\mathcal{E} \otimes 1)(\hat{\Delta}_q \otimes \iota)(\mathcal{E}).$$

Therefore $\mathcal{E}$ is a 2-cocycle for $(\mathcal{U}(G_q), \hat{\Delta}_q)$. Since

$$\mathcal{E}\hat{\Delta}_q(\cdot)\mathcal{E}^{-1} = \mathcal{E}\mathcal{F}\hat{\Delta}(\cdot)\mathcal{F}^{-1}\mathcal{E}^{-1} = \mathcal{F}'\hat{\Delta}(\cdot)\mathcal{F}'^{-1} = \hat{\Delta}_q,$$

the cocycle $\mathcal{E} \in \mathcal{U}(G_q \times G_q)$ is invariant, and since

$$\mathcal{E}_{21}\mathcal{R}_\hbar \mathcal{E}^{-1} = \mathcal{E}_{21}\mathcal{F}_{21}q^t \mathcal{F}^{-1}\mathcal{E}^{-1} = \mathcal{F}'_{21}q^t \mathcal{F}'^{-1} = \mathcal{R}_\hbar,$$

it is symmetric. By Theorem 2.1 there exists a central element c of $\mathcal{U}(G_q) = \mathcal{U}(G)$ such that

$$\mathcal{E} = (c \otimes c)\hat{\Delta}_q(c)^{-1},$$

so that $\mathcal{F}' = (c \otimes c)\hat{\Delta}_q(c^{-1})\mathcal{F} = (c \otimes c)\mathcal{F}\hat{\Delta}(c^{-1})$, and the first claim is proved. The second claim is immediate from Lemma 1.1. $\qquad\square$

As for dependence on φ, if $\varphi': \mathcal{U}(G_q) \to \mathcal{U}(G)$ is another isomorphism extending the canonical identification of the centers, there exists an invertible element u of $\mathcal{U}(G)$ such that $\varphi' = u\varphi(\cdot)u^{-1}$, and then $\mathcal{F}_u = (u \otimes u)\mathcal{F}\hat{\Delta}(u)^{-1}$ is a Drinfeld twist for φ'. By Theorem 5.2 all Drinfeld twists for φ' will therefore be cohomologous to $\mathcal{F}_u$. So up to coboundary, there is only one Drinfeld twist irrespectively of the choice of the isomorphism φ. When one considers a $*$-isomorphism φ together with a unitary Drinfeld twist $\mathcal{F}$, then u can be chosen to be unitary, and consequently, irrespectively of φ, there is only one unitary Drinfeld twist up to coboundary of a unitary element.

In the language of cohomology from Section 1, the Drinfeld associator $\Phi = \Phi_{KZ}$ is a unitary counital invariant 3-cocycle for $(\mathcal{U}(G), \hat{\Delta})$ satisfying the equation

$$\mathcal{R}_{12}\Phi_{312}\mathcal{R}_{13}\Phi_{132}^{-1}\mathcal{R}_{23}\Phi_{123} = \Phi_{321}\mathcal{R}_{23}\Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{213}\mathcal{R}_{12},$$

which is some sort of symmetry condition. Theorem 5.1 tells us then that $\Phi = \partial(\mathcal{F}^{-1})$, where the coboundary operator ∂ refers to $(\mathcal{U}(G), \mathcal{F}\hat{\Delta}(\cdot)\mathcal{F}^{-1})$, which is isomorphic to $(\mathcal{U}(G_q), \hat{\Delta}_q)$. This should be compared with Theorem 2.1 stating that any symmetric invariant 2-cocycle for $(\mathcal{U}(G_q), \hat{\Delta}_q)$ is the coboundary of a central element.

6. UNIQUENESS OF THE DIRAC OPERATOR

As in the previous section, we assume that $q > 0$ and $\hbar \in i\mathbb{R}$ is such that $q = e^{\pi i \hbar}$. In [12] we constructed a quantum Dirac operator D_q on G_q that defines a biequivariant spectral triple which is an isospectral deformation of that defined by the Dirac operator D on G. We briefly recall this construction.

The Riemannian metric on G is defined using the invariant form $-(\cdot, \cdot)$ on $\mathfrak{g}$. Consider a basis $\{x_i\}_i$ of $\mathfrak{g}$ such that $(x_i, x_j) = -\delta_{ij}$, and let $\gamma: \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ denote the inclusion of $\mathfrak{g}$ into the complex Clifford algebra with the convention that $\gamma(x_i)^2 = -1$. Identifying $\mathfrak{so}(\mathfrak{g})$ with $\mathfrak{spin}(\mathfrak{g})$, the adjoint action is defined by the representation $\widetilde{\mathrm{ad}}: \mathfrak{g} \to \mathfrak{spin}(\mathfrak{g}) \subset \mathrm{Cl}(\mathfrak{g})$ given by

$$\widetilde{\mathrm{ad}}(x) = \frac{1}{4}\sum_i \gamma(x_i)\gamma([x, x_i]).$$

We denote by the same symbol $\widetilde{\mathrm{ad}}$ the corresponding homomorphism $\mathcal{U}(G) \to \mathrm{Cl}(\mathfrak{g})$.

Let $s\colon \mathrm{Cl}(\mathfrak{g}) \to \mathrm{End}(\mathbb{S})$ be an irreducible representation. Denote by ∂ the representation of $U\mathfrak{g}$ by left-invariant differential operators. Identifying the sections $\Gamma(S)$ of the spin bundle S over G with $C^\infty(G) \otimes \mathbb{S}$, the Dirac operator $D\colon C^\infty(G) \otimes \mathbb{S} \to C^\infty(G) \otimes \mathbb{S}$ defined using the Levi-Civita connection, can be written as $D = (\partial \otimes s)(\mathcal{D})$, where $\mathcal{D} \in U\mathfrak{g} \otimes \mathrm{Cl}(\mathfrak{g})$ is given by the formula

$$\mathcal{D} = \sum_i (x_i \otimes \gamma(x_i) + \frac{1}{2} \otimes \gamma(x_i)\widetilde{\mathrm{ad}}(x_i)).$$

Denote by $\mathbb{C}[G_q]$ the linear span of matrix coefficients of finite dimensional admissible representations of $U_q\mathfrak{g}$. It is a Hopf $*$-algebra with comultiplication Δ_q, and $\mathcal{U}(G_q)$ is its dual space. Let $(L^2(G_q), \pi_{r,q}, \xi_q)$ be the GNS-triple defined by the Haar state on $\mathbb{C}[G_q]$. The left and right regular representations of $W^*(G_q)$ on $L^2(G_q)$, denoted by $\hat{\pi}_{r,q}$ and ∂_q correspondingly, are defined by

$$\hat{\pi}_{r,q}(\omega)\pi_{r,q}(a)\xi_q = (\omega S_q^{-1} \otimes \pi_{r,q})\Delta_q(a)\xi_q,$$

where S_q is the antipode on $\mathbb{C}[G_q]$, and

$$\partial_q(\omega)\pi_{r,q}(a)\xi_q = (\pi_{r,q} \otimes \omega)\Delta_q(a)\xi_q = a_{(1)}(\omega)\pi_{r,q}(a_{(0)})\xi_q.$$

Pick a $*$-isomorphism $\varphi\colon \mathcal{U}(G_q) \to \mathcal{U}(G)$ and a unitary Drinfeld twist $\mathcal{F} \in \mathcal{U}(G \times G)$. The quantum Dirac operator D_q is the unbounded operator on $L^2(G_q) \otimes \mathbb{S}$ defined by

$$D_q = (\partial_q \otimes s)(\mathcal{D}_q),$$

where $\mathcal{D}_q \in \mathcal{U}(G_q) \otimes \mathrm{Cl}(\mathfrak{g})$ is given by

$$\mathcal{D}_q = (\varphi^{-1} \otimes \iota)((\iota \otimes \widetilde{\mathrm{ad}})(\mathcal{F})\mathcal{D}(\iota \otimes \widetilde{\mathrm{ad}})(\mathcal{F}^*)).$$

The operator D_q is G_q-biequivariant in the sense that it commutes with all operators of the form $\hat{\pi}_{r,q}(x) \otimes 1$ and $(\partial_q \times s\,\widetilde{\mathrm{ad}}_q)(x)$, $x \in W^*(G_q)$, where $\widetilde{\mathrm{ad}}_q = \widetilde{\mathrm{ad}}\varphi$.

Theorem 6.1.

(i) *For fixed φ the Dirac operator D_q does not depend on the chosen Drinfeld twist $\mathcal{F}$.*

(ii) *The biequivariant spectral triple $(\mathbb{C}[G_q], L^2(G_q) \otimes \mathbb{S}, D_q)$ does not depend on the choice of φ and $\mathcal{F}$ up to unitary equivalence.*

Proof. By Theorem 5.2 any other unitary Drinfeld twist $\tilde{\mathcal{F}}$ for the same φ has the form

$$\tilde{\mathcal{F}} = (c \otimes c)\mathcal{F}\hat{\Delta}(c)^*$$

for a central unitary element c of $\mathcal{U}(G)$. Denoting the element $\mathcal{D}_q$ defined by $\tilde{\mathcal{F}}$ by $\tilde{\mathcal{D}}_q$, we get

$$\begin{aligned}
(\varphi \otimes \iota)(\tilde{\mathcal{D}}_q) &= (\iota \otimes \widetilde{\mathrm{ad}})((c \otimes c)\mathcal{F}\hat{\Delta}(c)^*)\mathcal{D}(\iota \otimes \widetilde{\mathrm{ad}})(\hat{\Delta}(c)\mathcal{F}^*(c^* \otimes c^*)) \\
&= (\iota \otimes \widetilde{\mathrm{ad}})((c \otimes c)\mathcal{F})\mathcal{D}(\iota \otimes \widetilde{\mathrm{ad}})(\mathcal{F}^*(c^* \otimes c^*)) && \text{(since } \mathcal{D} \text{ is } \mathfrak{g}\text{-invariant)} \\
&= (1 \otimes \widetilde{\mathrm{ad}}(c))(\iota \otimes \widetilde{\mathrm{ad}})(\mathcal{F})\mathcal{D}(\iota \otimes \widetilde{\mathrm{ad}})(\mathcal{F}^*)(1 \otimes \widetilde{\mathrm{ad}}(c^*)) && \text{(since } c \text{ is central)} \\
&= (\iota \otimes \widetilde{\mathrm{ad}})(\mathcal{F})\mathcal{D}(\iota \otimes \widetilde{\mathrm{ad}})(\mathcal{F}^*) && \text{(since } \widetilde{\mathrm{ad}}(c) \text{ is a scalar),}
\end{aligned}$$

where in the last step we used the known fact that $\widetilde{\mathrm{ad}}$ is a multiple of an irreducible representation, namely, of the representation with highest weight ρ, half the sum of positive roots. This proves (i).

If we choose another $*$-isomorphism $\varphi'\colon \mathcal{U}(G_q) \to \mathcal{U}(G)$, there exists a unitary u such that $\varphi' = u\varphi(\cdot)u^*$. We can take the element $\mathcal{F}' = (u \otimes u)\mathcal{F}\hat{\Delta}(u^*)$ as a unitary Drinfeld twist for φ'. Then for the element $\mathcal{D}'_q$ defined by φ' and $\mathcal{F}'$ we get, using that $\mathcal{D}$ commutes with $(\iota \otimes \widetilde{\mathrm{ad}})\hat{\Delta}(u)$, that

$$\mathcal{D}'_q = (\varphi'^{-1} \otimes \iota)\left((\iota \otimes \widetilde{\mathrm{ad}})((u \otimes u)\mathcal{F})\mathcal{D}(\iota \otimes \widetilde{\mathrm{ad}})(\mathcal{F}^*(u^* \otimes u^*))\right) = (1 \otimes \widetilde{\mathrm{ad}}(u))\mathcal{D}_q(1 \otimes \widetilde{\mathrm{ad}}(u^*)).$$

We also have $\widetilde{\mathrm{ad}}_q' := \widetilde{\mathrm{ad}}\varphi' = \widetilde{\mathrm{ad}}(u)\widetilde{\mathrm{ad}}_q(\cdot)\widetilde{\mathrm{ad}}(u^*)$. Therefore the operator $1 \otimes s\,\widetilde{\mathrm{ad}}(u)$ provides a unitary equivalence between the biequivariant spectral triples $(\mathbb{C}[G_q], L^2(G_q) \otimes \mathbb{S}, D_q)$ and $(\mathbb{C}[G_q], L^2(G_q) \otimes \mathbb{S}, D_q')$. $\qquad\qquad\square$

APPENDIX A.

Let G be a compact Lie group and $G_\mathbb{C}$ its analytic complexification. By definition, any continuous finite dimensional representation $G \to \mathrm{GL}(V)$ extends uniquely to a holomorphic representation $G_\mathbb{C} \to \mathrm{GL}(V)$. Hence every element $g \in G_\mathbb{C}$ can be considered as an element of $\mathcal{U}(G)$. Furthermore, $\hat{\Delta}(g) = g \otimes g$ by analyticity, since this is true for all $g \in G$ (to be more precise, in order to not worry about topology, we should first apply a finite dimensional representation $\pi_1 \otimes \pi_2$). Therefore $G_\mathbb{C}$ consists of group-like elements. We will show that these are all; for $G = \mathrm{SU}(n)$ this is [1, Theorem 2].

Theorem A.1. *For any compact Lie group G the set of group-like elements in $\mathcal{U}(G)$ coincides with $G_\mathbb{C}$.*

Proof. Let $a \in \mathcal{U}(G)$ be a group-like element, so a is invertible and $\hat{\Delta}(a) = a \otimes a$. Assume first that a is bounded, so it belongs to the von Neumann algebra $W^*(G) \subset \mathcal{U}(G)$ of G. Then $a \in G \subset G_\mathbb{C}$. This is a well-known result going back to Tatsuuma [14] and valid for any locally compact group. Here is a short proof.

Consider $W^*(G)$ as the von Neumann algebra generated by the operators λ_g of the left regular representation of G. Therefore we want to prove that $a = \lambda_g$ for some $g \in G$. Let U be an open neighbourhood of the unit element $e \in G$. Consider the set K_U consisting of all elements $g \in G$ for which there exists a function $f \in L^2(G)$ with essential support in U such that $(\mathrm{ess.\,supp}\,af) \cap gU \neq \emptyset$. As the whole space $L^2(G)$ is spanned by right translations of functions with essential support in U, there exists f with $\mathrm{ess.\,supp}\,f \subset U$ such that $af \neq 0$. It follows that K_U is non-empty. We claim that if $g_0 \in K_U$ then

(i) the element a lies in the strong operator closure of the span of λ_g with $g \in g_0 UU^{-1}$;

(ii) $K_U \subset \overline{g_0 UU^{-1}UU^{-1}}$.

Indeed, consider functions f and h such that $\mathrm{ess.\,supp}\,f \subset U$, $\mathrm{ess.\,supp}\,h \subset g_0 U$ and $(af, h) \neq 0$. Denote by ω the normal linear functional $(\cdot\,f, h)$ on $B(L^2(G))$. Then

$$(\iota \otimes \omega)\hat{\Delta}(a) = (\iota \otimes \omega)(a \otimes a) = \omega(a)a,$$

and on the other hand,

$$(\iota \otimes \omega)\hat{\Delta}(\lambda_g) = \omega(\lambda_g)\lambda_g = 0 \quad \text{for} \quad g \notin g_0 UU^{-1}.$$

Since a can be approximated by linear combinations of the operators λ_g, applying the normal operator $(\iota \otimes \omega)\hat{\Delta}$ to these approximations we get (i). Now if $f \in L^2(G)$ is arbitrary with $\mathrm{ess.\,supp}\,f \subset U$, by (i) we have $\mathrm{ess.\,supp}\,af \subset \overline{g_0 UU^{-1}U}$, whence $K_U \subset \overline{g_0 UU^{-1}UU^{-1}}$.

If $V \subset U$ are two neighbourhoods of $e \in G$ then clearly $K_V \subset K_U$. Property (ii) implies that the intersection of the sets $\overline{K}_U$ consists of exactly one point, which we denote by g_0. Property (i) implies that a belongs to the strong operator closure of the span of the operators λ_g with g lying in an arbitrarily small neighbourhood of g_0. We want to prove that this forces $a = \lambda_{g_0}$. Replacing a by $\lambda_{g_0}^{-1}a$ we may assume that $g_0 = e$. Then for any $f \in L^2(G)$ we get

$$\mathrm{ess.\,supp}\,af \subset \mathrm{ess.\,supp}\,f.$$

If we consider the action of $L^\infty(G)$ on $L^2(G)$ by multiplication, by regularity of the Haar measure this implies that a commutes with the characteristic function of any measurable set. It follows that $a \in L^\infty(G)$. Since a commutes with the operators of the right regular representation, this implies that a is a scalar, and since it is group-like, we get $a = 1$.

Consider now an arbitrary group-like element $a \in \mathcal{U}(G)$. Then $a^* a$ is group-like as well, hence $|a|$ is also group-like. It follows that if $a = u|a|$ is the polar decomposition then u is group-like. By the first part of the proof we know that $u \in G$. So we just have to show that $|a| \in G_{\mathbb{C}}$. In other words, we may assume that a is positive.

For every $z \in \mathbb{C}$ we have

$$\hat{\Delta}(a^z) = \hat{\Delta}(a)^z = (a \otimes a)^z = a^z \otimes a^z.$$

In particular, the bounded elements a^{it}, $t \in \mathbb{R}$, are group-like, hence they lie in $G \subset \mathcal{U}(G)$. It follows that there exists $X \in \mathfrak{g}$ such that $a^{it} = \exp tX$ for $t \in \mathbb{R}$, whence $a^z = \exp(-izX) \in G_{\mathbb{C}}$ for all $z \in \mathbb{C}$, since both a^z and $\exp(-izX)$ are analytic functions in z which coincide for $z \in i\mathbb{R}$. In particular, $a = \exp(-iX) \in G_{\mathbb{C}}$. $\qquad\square$

Appendix B.

The proof of the main theorem can also be applied in the formal deformation setting. However, in this case it is easier to follow Drinfeld's cohomological arguments for 3-cocycles [6], see also the proof of [7, Theorem XVIII.8.1]. Although a translation of those arguments into our setting of 2-cocycles is completely straightforward, we include it in this appendix for the reader's convenience.

Theorem B.1. *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra, and let $(U\mathfrak{g}[[h]], \hat{\Delta}_h, \mathcal{R}_h)$ be a quasitriangular deformation of $(U\mathfrak{g}, \hat{\Delta})$. Assume $\mathcal{E} \in (U\mathfrak{g} \otimes U\mathfrak{g})[[h]]$ is a symmetric invariant 2-cocycle such that $\mathcal{E} = 1 \mod h$, so*

(i) $[\mathcal{E}, \hat{\Delta}_h(a)] = 0$ *for all $a \in U\mathfrak{g}[[h]]$;*

(ii) $\mathcal{R}_h \mathcal{E} = \mathcal{E}_{21} \mathcal{R}_h$;

(iii) $(\mathcal{E} \otimes 1)(\hat{\Delta}_h \otimes \iota)(\mathcal{E}) = (1 \otimes \mathcal{E})(\iota \otimes \hat{\Delta}_h)(\mathcal{E})$.

Then there exists a central element $c \in U\mathfrak{g}[[h]]$ such that $c = 1 \mod h$ and $\mathcal{E} = (c \otimes c)\hat{\Delta}_h(c)^{-1}$.

Proof. We will construct by induction central elements $c_n \in U\mathfrak{g}[[h]]$, $n \geq 0$, such that $c_0 = 1$ and

$$\mathcal{E} = (c_n \otimes c_n)\hat{\Delta}_h(c_n)^{-1} \mod h^{n+1} \quad \text{and} \quad c_n = c_{n-1} \mod h^n \quad \text{for} \quad n \geq 1.$$

Then the sequence $\{c_n\}_n$ converges to the required element c.

Assume $c_0, \ldots, c_{n-1}$ are constructed. Let $\varphi \in U\mathfrak{g} \otimes U\mathfrak{g}$ be such that

$$\mathcal{E} = (c_{n-1} \otimes c_{n-1})\hat{\Delta}_h(c_{n-1})^{-1} + h^n \varphi \mod h^{n+1}.$$

Reducing conditions (i)-(iii) modulo h^{n+1} and using that $\hat{\Delta}_h = \hat{\Delta}$, $\mathcal{R}_h = 1$ and $c_{n-1} = 1$ modulo h, we get

$$[\varphi, \hat{\Delta}(a)] = 0 \text{ for } a \in U\mathfrak{g}, \quad \varphi = \varphi_{21} \quad \text{and} \quad \varphi \otimes 1 + (\hat{\Delta} \otimes \iota)(\varphi) = 1 \otimes \varphi + (\iota \otimes \hat{\Delta})(\varphi).$$

In the notation of [7, Ch. XVIII.5] the last two identities mean that φ is a 2-cocycle in the complex $(T_-(U\mathfrak{g}), \delta)$. Since the symmetrization map $\eta \colon S\mathfrak{g} \to U\mathfrak{g}$ is an isomorphism of coalgebras, we have $H^{2k}(T_-(U\mathfrak{g}), \delta) = 0$ for all $k \geq 0$ by [7, Theorem XVIII.7.1]. Therefore φ is the coboundary of an element $f \in U\mathfrak{g}$, so that

$$\varphi = f \otimes 1 + 1 \otimes f - \hat{\Delta}(f).$$

Furthermore, since φ is $\mathfrak{g}$-invariant, by [7, Proposition XVIII.6.2] we can choose f to be $\mathfrak{g}$-invariant as well. Then we put $c_n = (1 + h^n f)c_{n-1}$. $\qquad\square$

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